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Convergence properties of the functional self-organization stochastic algorithm

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Abstract. We investigate the convergence properties of the functional self-organization algorithm. We explicitly establish the necessary conditions to start the most simple global coherent task—clusters (piles) aggregation. Based on the intermediate steady states assumption and numerical results we show that the growth probability of the same type of clusters are proportional to the clusters' dimensions.

1. Introduction

Many natural systems display a complex collective behaviour. The global spatio-temporal behaviour of a system consisting of a large number of individual units, viewed as dynamical systems in themselves, interacting with each other, is usually termed *swarm intelligence*. The paradigm of complexity consists of a complex spatio-temporal behaviour that emerges from relatively simple local rules. Synchronization is one of the most important phenomena observed in biological, chemical and physical systems. The mechanism of collective synchronous behaviour at play, even in simple models such as Chate–Maneville (CM) cellular automata (CA) or the Bak, Tang and Wiesenfeld (BTW) model of critically self-organized (CSO) systems, remains elusive and satisfactory analytical solutions are still missing [1, 8, 12, 14].

Starting from experimental studies and numerical simulations done by Deneubourg and co-workers [2–7] a stochastic nonlinear model of functional self-organization (FSO) [9, 10] was developed. We prefer the term FSO (introduced by Deneubourg) instead of *swarm intelligence* for its generality. This study refers to the particular problem of the convergence of a new algorithm proposed to simulate FSO processes. In our previous papers a CA model for the FSO procesess [9, 10] was proposed. The FSO model is based on three concepts: *environment*, *entity* and *object*. The *environment*, usually, but not necessarily, is a two-dimensional lattice. Throughout the *environment* some *entities* (robot-like ants (RLA) according to Deneubourg) perform a random walk motion. At the lattice sites there are different *object types*, denoted by $a, b, c, \ldots$ (the empty site is a special *object*, ϕ). Every *entity* recognizes the *object's type*, can pick up an *object*, transport and put it down on an empty lattice site. The *entities* have no map of the *environment* and there is no direct communication between them. One of the most simple global tasks for the system consists of a cluster (pile) of the same *object types'* aggregation. To obtain a global emergent behaviour every *RLA* has *memory* registers. Thus, to every encountered *object*

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type there corresponds a binary record in the memory of the *entity*. The current binary record corresponding to the *α-type object* receives '1' if the current encountered *object* is an *α-type object*, otherwise receives '0'. At each timestep the *RLA* carries an *object* (a free *RLA* carries a *φ-type object*) and has to decide whether to swap the carried *object* with the one it has. The swapping probability depends on how many *objects* of the same type the *RLA* has met in the past. From this viewpoint, this model seems to be of CM type due to the usage of the totalistic local rule. The major difference consists of the usage of the totalistic rule over the memorized record, namely, over the stochastic path.

The applications of the FSO model range from the *classical* study of the social behaviour of ant colonies [2–7], robot team coordination [13], immune system response [11], double-layer plasma self-organization, synthetic multitexture generation and image processing [9, 10], etc.

This paper is organized as follows. In section 2 the model of FSO is defined and principal concepts are explained. In section 3 we investigate the necessary convergence conditions to start cluster aggregation. Numerical results to support our theoretical findings are provided. In section 4 we present and discuss sufficient convergence conditions. Finally, in section 5 we provide a short summary of the results and current progress.

2. The model

The basic assumptions of our FSO model follow.

(1) The *environment* is a rectangular two-dimensional lattice with $N_x \times N_y$ sites. The lattice sites are occupied by *objects*, denoted by the letters $a, b, c, \ldots$. A free site is said to be occupied by an *φ-type object*.

(2) At any moment any *entity (robot, RLA)* carries an *object*. The *RLA* performs a random walk through the lattice. At each timestep only one *RLA* can be in a given lattice site.

(3) When a *RLA* moves to a given site it must decide whether or not to put down the carried *object* and to pick up the existing one.

The swapping condition reads as

$$f_\alpha > f_\beta \tag{1}$$

where f_α is the *weighted frequency* of the carried *α-type object* and f_β is the *weighted frequency* of the encountered *β-type object*.

If the *goal* is to form only simple patterns, such as clusters (piles) of the same *object type*, then the local decision (1) is similar to the CM [9, 10] totalistic rule. The only difference is that we apply a totalistic path rule instead of a neighbourhood orientated one. Therefore, according to the totalistic path rule, or memory based one, if the carried *object* has been met more frequently than the encountered one then the *RLA* decides to swap the two *objects*. Let us refer to the following memorized string

$$\begin{array}{lll} & \Downarrow & \text{the most recent entry in the memory} \\ s_\tau: & b & b\ b\ b\ b\ a\ a\ a\ a\ a \end{array} \tag{2}$$

which shows that the *RLA* has met five *a-type objects* and five *b-type objects* in the past. In order to simplify numerical evaluations every *RLA* associates a binary string to every *object type*. Therefore, using the string (2) as an example, the corresponding binary string for the *a-type object* is

$$s_a : 0000011111. \tag{3}$$

Generally speaking, a binary string of length τ has the following form

$$s_{\alpha,\tau} : u_{\alpha,1}u_{\alpha,2}\ldots u_{\alpha,\tau} \tag{4}$$

where

$$u_{\alpha,i} = \begin{cases} 1 & \text{if an } \alpha\text{-}\textit{type object} \text{ was encountered at step } i \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

To make a decision according to the previously mentioned global goal (clusters of identical *objects* aggregation) every *object type* is characterized, at any instant, by a weighted frequency

$$f_\alpha(\tau) = \frac{\sum_{i=1}^{\tau} w(i)u_{\alpha,i}}{\sum_{i=1}^{\tau} w(i)} \tag{6}$$

where $w(i)$ is an appropriate *weight function*. To overwhelm the old troubleshooting of the original FSO mechanism [2–7] we used, for the first time, a recursive defined memory that allows a whole history record. *The weighting function* has the form

$$w(i) = \frac{1}{r^{i-1}} \tag{7}$$

with r a positive parameter—*the memory radius*. From (7) and (6) one obtains

$$f_\alpha(K) = r^{K-1}\frac{r-1}{r^K-1}\sum_{i=1}^{K}\frac{u_{\alpha,i}}{r^{i-1}}. \tag{8}$$

From (8) it can be seen that if $r \gg 1$, then the contribution of the τth step (with $\tau \gg 1$) to the current decision is quite insignificant. Therefore, we may say that only the most recent steps contribute to the decision or, in other words, that we have a *short memory*. The limiting case $r = 1$ corresponds to an infinite memory. The case when $r < 1$ exacerbates the contributions of the τ steps with $\tau \gg 1$ and diminishes the contributions of the most recent ones. In the present simulations $r > 1$.

Let us first observe that each time we compare two *weighted frequencies* the factor behind the sum in (8) is the same and can be always omitted. Therefore, to make a decision it is sufficient to compute, at any instant, the sum

$$S_\alpha^n = \sum_{i=1}^{\tau}\left(u_{\alpha,i}\right)_n w(i) \tag{9}$$

where $(u_{\alpha,i})_n$ is the binary digit corresponding to the ith place in the α string at the nth iteration step. The next step requires a new evaluation of the sum

$$S_\alpha^{n+1} = \sum_{i=1}^{\tau+1}(u_{\alpha,i})_{n+1}w(i) \tag{10}$$

where the following shifting rule takes place

$$\begin{aligned} &(u_{\alpha,i})_{n+1} = (u_{\alpha,i-1})_n \qquad \text{for } i \geqslant 2 \\ &(u_{\alpha,1})_{n+1} = \begin{cases} 1 & \text{if the object left in the current site is of } \alpha\text{-}\textit{type} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{11}$$

Using (11), (10) and (9) one obtains

$$S_\alpha^{n+1} - S_\alpha^n = (u_{\alpha,1})_{n+1}w(1) + \frac{1-r}{r}S_\alpha^n \tag{12}$$

which is the Langevin stochastic map of the process. It is obvious from (12) that there is no loss of information using the proposed form (7) of *the weighted memory function*. Moreover, to compute and compare the two sums at a give instant we need only two memory cells for every *object type*. Despite its simplicity, the first-order recurrent scheme (12) does not allows a truly infinite record. This limitation follows from the internal finite-digit representation of any real number [9, 10]. This fact leads to an *effective memory length* τ greater than that used in [2–7] but finite (see [9, 10]).

3. The necessary conditions for the convergence of the FSO algorithm

First, we demonstrate the necessary convergence condition in order to start to aggregate clusters of the same *objects type* from an initially random distribution.

Here, and throughout the whole paper, we mean by *convergence of the FSO algorithm* the property of the locally oriented dynamics to drive the global behaviour to the *desired macroscopic behaviour (e.g. clusters (piles) aggregation)*.

For this purpose let us assume, without any loss of generality, that at the first step a *RLA* had met an *a-type object* and picked it up. Then let us suppose that the *RLA* had moved through a field entirely occupied by *b-type objects* and, after τ steps (*the effective memory length* [9]), it had again met an *a-type object*.

The less favourable string for the *a-type object* aggregation is

$$s_{a,\tau} : \overbrace{100\ldots0}^{\tau} \tag{13}$$

where the last entry indicates that the *RLA* had previously met only one *a-type object*. The corresponding string for the *b-type objects* is

$$s_{b,\tau} : \overbrace{011\ldots1}^{\tau}. \tag{14}$$

The *necessary (convergence) condition* to put down the carried *object* in one of the four neighbour lattice sites of the most recent *a-type object* encountered is $f_a(\tau) > f_b(\tau)$ which, using (6), can be written

$$r^{\tau+1} - 2r^{\tau} + 1 > 0.$$

This relation is satisfied if and only if $r \in (r_0(\tau), 2)$, where $r_0(\tau)$ is the root of the equation

$$r^{\tau+1} - 2r^{\tau} + 1 = 0$$

with $r \in (1, 2)$. Therefore, even in the less favourable case for the two-object cluster aggregation, it is possible to choose r such that a *two-object cluster* appears. Once a *two-object cluster* has been formed it starts to grow [9]. On the other hand, the algorithm *converges* for any concentration of the *objects* [9, 10]. This is one of the most important achievements, which does not work in [2–7].

How long will it take to reach the less probable configuration or, equivalently, what is the probability for the associated Markov chain (13) to occur? To estimate this probability we used a mean-field viewpoint. Let us suppose that the probability of finding an *object* in a given site does not depends on the other *objects'* positions. Therefore, the probability of finding an *object*, say of *a-type*, can be approximated in the neighbourhood of an arbitrary lattice site by $p_a = c_a \frac{v}{V} = c_a \frac{4}{V}$, where $c_a = \frac{N_a}{N}$ is the concentration of the *a-type objects*, v is the *volume* of the neighbourhood and V is the lattice *volume*. Let $p_{a,i}$ be the probability

of finding i ($4 \geqslant i \geqslant 0$) *a-type objects* in the neighbourhood of an arbitrary site. An approximation of this probability is given by the binomial distribution

$$p_{a,i} = \binom{i}{N_a} p_a^i (1 - p_a)^{N_a - i}.$$

The mean number of the *a-type objects* in the neighbourhood of an arbitrary lattice site reads as

$$\bar{\xi}_a = \frac{\sum_{i=0}^{4} i p_{a,i}}{\sum_{i=0}^{4} p_{a,i}} = \frac{\sum_{i=0}^{4} i \binom{i}{N_a} p_a^i (1 - p_a)^{N_a - i}}{\sum_{i=0}^{4} \binom{i}{N_a} p_a^i (1 - p_a)^{N_a - i}}. \tag{15}$$

The above relation allows to write the mean probability of finding an *a-type object* in the neighbourhood of an arbitrary lattice site in the form

$$\bar{p}_a = \tfrac{1}{4}\bar{\xi}_a$$

and for the *b-type object*

$$\bar{p}_b = \tfrac{1}{4}\bar{\xi}_b = \tfrac{1}{4}(4 - \bar{\xi}_b) = 1 - \bar{p}_b.$$

In order to obtain the probability to realize the less favourable path the following assumptions were made:

- jumps are independent events (random walk),
- the *RLA* met at the first step an *a-type object* with the probability $\frac{N_a}{N}$ and picked it up,
- the *robot* moves $\tau - 2$ steps through a *b-type objects* field,
- finally, the *RLA* met another *a-type object*.

Using the above conditions, one may write

$$p_{\min} = \frac{N_a}{N}(1 - \bar{p}_a)^{\tau - 2}\bar{p}_a \tag{16}$$

where the first factor is taken because the *RLA* met an *a-type object* at the first step with the probability $\frac{N_a}{N}$ and that the *object* is then carried $\tau - 1$ steps until another *a-type object* is found. It follows that the maximum number of steps which ensures convergence is

$$\mathcal{N}_{\max} = \frac{1}{p_{\min}} = \frac{N}{N_a}\frac{1}{\bar{p}_a(1 - \bar{p}_a)^{\tau - 2}}. \tag{17}$$

The influence of the concentration c of the *objects* and the *effective memory length* τ on the conventional Monte Carlo simulation timesteps is shown in figure 1.

Our numerical simulations agree with the theoretically evaluated number of steps (17) (see figure 2).

It is also intuitively obvious that the *memory radius* must depend on the cluster dimension. Therefore, to obtain an optimum computational time we have to perform a *simulated annealing* with respect to the *memory radius*. The work is in progress and the results will be published in a forthcoming paper.

4. The sufficient convergence conditions

The above results refer only to *the necessary convergence conditions*. We imagined that the system's global behaviour is organized such that, starting with a random distribution of *objects* through the *environment*, the *RLA* first organize *two-object clusters* over the whole lattice, then *three-object clusters* and so on. The advantage of this *intermediate steady states behaviour* is an analytical realistic prediction on the system's global behaviour. On

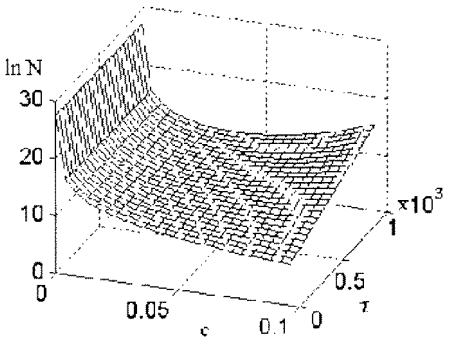

Figure 1. The plot of the conventional Monte Carlo timesteps dependence on concentration c and *effective memory length* τ using logarithmic coordinates is shown. The values of the effective memory length τ correspond to the memory radius $\in (1, 2)$.

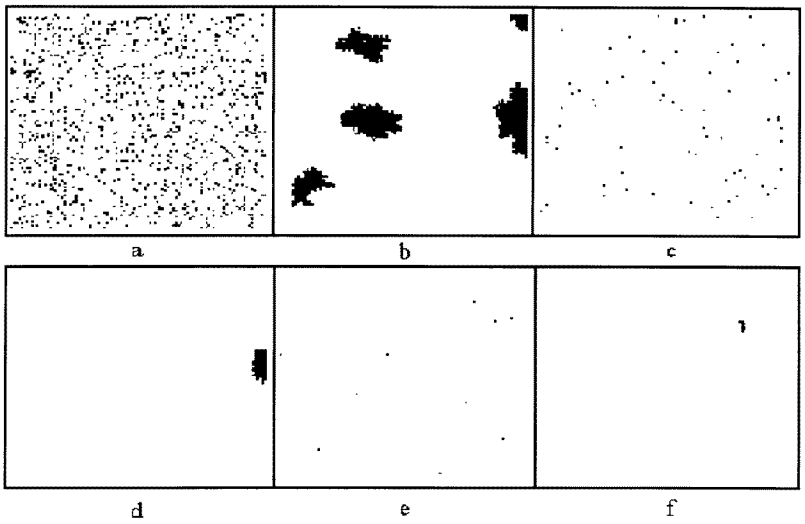

Figure 2. The picture of two-type clusters aggregation in a two-dimensional rectangular 100×100 lattice with 30 *RLA* and a *memory radius* $r = 1.05$. The initial configurations ((*a*), (*c*), respectively (*e*)) are randomly generated with 10%, 1% and, respectively, 0.1% concentration of the *a-type* objects (black pixels). The final aggregation stages were obtained after (*b*) 10^6 steps, (*d*) 0.5×10^6 steps and (*f*) 0.9×10^6 steps.

the basis of *the intermediate steady states* assumption we found necessary conditions to form *two-object clusters*, *three-object clusters*, etc. As a drawback of the above assumption we may observe that the clusters' (piles') aggregation is a competitive growing process and therefore *the necessary convergence conditions* may not be *sufficient* in order to reach a final steady state of the aggregation process.

In this section we deal with *sufficient condition* for cluster aggregation.

Let $P_i(x, t)$ be the probability for the ith *RLA* to be in the site $x \in \{1, 2, \ldots, N\}$ at timestep t. In the following we will consider only one RLA in the lattice. Then the master equation of the process can be written

$$P_i(x, t+1) = P_i(x-1, t)p^+ + P_i(x+1, t)p^- \tag{18}$$

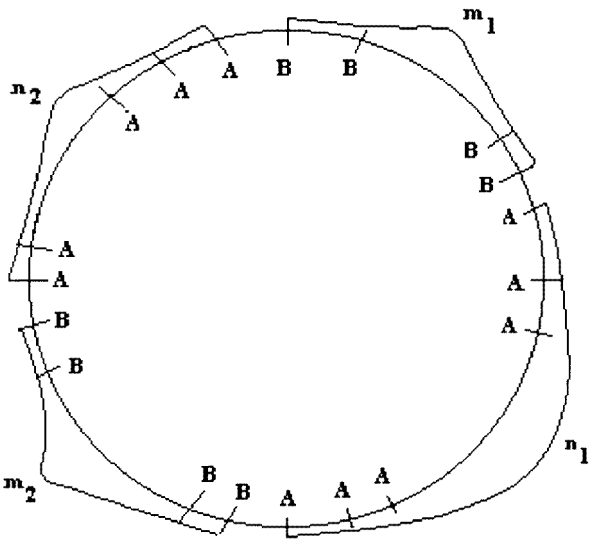

Figure 3. The one-dimensional periodic lattice with two clusters of *a-type objects* and two clusters of *b-type objects*.

where $p^+(p^-)$ is the right (left) transition probability. Let the vector

$$\boldsymbol{v}(t)^T = (P(1,t)P(2,t)\dots P(N,t)) \tag{19}$$

where the superior index T means transposition. Using vectorial notation (19), it is easy to cast equation (18) into the form

$$\boldsymbol{v}(t+1) = A\boldsymbol{v}(t) \tag{20}$$

where

$$A = \begin{bmatrix} 0 & p^- & 0 & \dots & 0 & p^+ \\ p^+ & 0 & p^- & \dots & 0 & 0 \\ 0 & p^+ & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & p^- \\ p^- & 0 & 0 & \dots & p^+ & 0 \end{bmatrix} \tag{21}$$

with the initial condition

$$\boldsymbol{v}(0)^T = (00\dots010\dots00) \tag{22}$$

if the *RLA* starts at timestep $t = 0$ from $x = i$. In the preceding section we showed (based on theoretical and numerical arguments) that the FSO algorithm starts to form clusters (piles) of objects selected by their type. But, near the final aggregation stage it is possible to obtain, for example, only two *a-type* clusters (piles). The *RLA* pick up *a-type objects* from one *a-type* clusters, transport and put them down in the other one. Next the *RLA* can decide to move an *object* back to the old cluster and so on. Therefore, we may ask: Can the algorithm be trapped in an infinite oscillatory loop? To demonstrate that our FSO algorithm does not allow oscillatory phenomena let us refer to the particular configuration from figure 3.

Using (8) it is straightforward to show that if $f_a(\tau) > f_b(\tau)$ for the memory radius $r = 1$ then it is possible to find $r_1 \geqslant r \geqslant 1$ such that $f_a(\tau) > f_b(\tau)$.

Indeed, let us consider two binary strings $u_{a,i}$ and $u_{b,i}$ with $i = 1, 2, \ldots, \tau$ which satisfy the condition $f_a(\tau) > f_b(\tau)$ for $r = 1$ namely, $\sum_{i=1}^{\tau} u_{a,i} = \alpha \geqslant \frac{\tau}{2}$. If $r > 1$ the swapping condition $f_a(\tau) > f_b(\tau)$ depends on r, τ and the statistical distribution of the binary digits $u_{a,i}$. We expect a typical occurrence for the binary strings of the form $s_a : \underbrace{\overbrace{11\ldots1}^{\lambda_1}\overbrace{00\ldots0}^{\lambda_2}\overbrace{11\ldots1}^{\lambda_1}\overbrace{00\ldots0}^{\lambda_2}\ldots}_{\tau}$ with λ_1 and λ_2 some characteristic lengths depending on the aggregation stage. If the swapping condition is satisfied for the less favourable binary string $s_a : \underbrace{\overbrace{00\ldots0}^{\tau-\alpha}\overbrace{11\ldots1}^{\alpha}}_{\tau}$ with $\alpha \geqslant \frac{\tau}{2}$ then that will be true for any binary string. Using (8) the swapping condition gives

$$r^{\tau} - 2r^{\tau-\alpha} + 1 < 0$$

namely, $r \in (1, r_1(\tau))$ where $r_1(\tau) < r_0(\tau)$ is the solution of the equation

$$r^{\tau} - 2r^{\tau-\alpha} + 1 = 0.$$

Therefore, there exists a limit value $r_1(\tau) > 1$ of the *memory radius* such that, for every $r \in (1, r_1(\tau))$, the condition $f_a(\tau) > f_b(\tau)$ is satisfied. That is why we refer in the following only to the convergence of the algorithm when $r = 1$. If in the lattice there are only two different *object types* then the condition $f_a(\tau) > f_b(\tau)$ is satisfied if the number of the *a-type objects* encountered is greater than for the *b-type objects*. Let $p_i(x)$ be the probability of finding an *a-type object* at the ith step, starting from an arbitrary lattice site x. The probability of realizing $f_a(\tau) > f_b(\tau)$ starting from x_0 is

$$\begin{aligned}
P(x_0, f_a > f_b) &= p_1(x_0)p_2(x_0)\ldots p_\tau(x_0) + p_1^*(x_0)p_2(x_0)\ldots p_\tau(x_0) \\
&\quad + p_1(x_0)p_2^*(x_0)\ldots p_\tau(x_0) + p_1(x_0)p_2(x_0)\ldots p_\tau^*(x_0) \\
&\quad + p_1^*(x_0)p_2^*(x_0)\ldots p_\tau(x_0) + \cdots = p_1(x_0)p_2(x_0)\ldots p_\tau(x_0) \\
&\quad \times\left[1 + \sum_{i_1>i_2}^{\tau} \frac{p_{i_1}^*(x_0)p_{i_2}^*(x_0)}{p_{i_1}(x_0)p_{i_2}(x_0)} + \sum_{i_1>i_2>i_3}^{\tau} \frac{p_{i_1}^*(x_0)p_{i_2}^*(x_0)p_{i_3}^*(x_0)}{p_{i_1}(x_0)p_{i_2}(x_0)p_{i_3}(x_0)}\right. \\
&\quad \left. + \cdots + \sum_{i_1>i_2>\ldots>i_K}^{\tau} \frac{p_{i_1}^*(x_0)p_{i_2}^*(x_0)\ldots p_{i_K}^*(x_0)}{p_{i_1}(x_0)p_{i_2}(x_0)\ldots p_{i_K}(x_0)}\right] \qquad (23)
\end{aligned}$$

with $K = [\frac{\tau}{2}] + 1$, where [ ] is the round function and $p_i^* = 1 - p_i$.

Equation (23) can be written

$$\begin{aligned}
P(x_0, f_a > f_b) &= p_1(x_0)p_2(x_0)\ldots p_\tau(x_0)\left[\sum_{j=0}^{[\frac{\tau}{2}]+1}(-1)^j\binom{j}{\tau}\right. \\
&\quad \left. + \Pi_1\sum_{j=0}^{[\frac{\tau}{2}]}(-1)^j\binom{j}{\tau} + \cdots + \Pi_{[\frac{\tau}{2}]+1}\sum_{j=0}^{0}(-1)^j\binom{j}{\tau}\right] \qquad (24)
\end{aligned}$$

where

$$\Pi_j = \sum_{i_1>i_2>\cdots>i_j}^{\tau} \frac{1}{p_{i_1}(x_0)p_{i_2}(x_0)\ldots p_{i_j}(x_0)} \qquad (25)$$

with $j = 1, 2, \ldots, [\frac{\tau}{2}] + 1$. The above equation can be expressed in a more compact form

$$P(x_0, f_a > f_b) = \Pi_\tau \sum_{i=0}^{K} \Pi_i \sum_{j=0}^{K-i}(-1)^j\binom{j}{\tau}. \qquad (26)$$

On the other hand, the probability of finding an *a-type object* at the ith step starting from an arbitrary lattice site x is given by the binomial distribution

$$p_i(x) = \sum_{j=0}^{i} \binom{j}{i} (p^+)^j (p^-)^{i-j} g(x - i + 2j) \tag{27}$$

where $g(x)$ is the characteristic function of the x site

$$g(x) = \begin{cases} 1 & \text{if there is an } \textit{a-type object} \text{ in the } x\text{th site} \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Let P_I be the probability of realizing $f_a > f_b$ when *RLA* is in the first *a-type* cluster, namely ($x \in \{1, 2, \ldots, n_1\}$), and P_{II} be the probability of realizing $f_a > f_b$ when *RLA* is in the second *a-type* cluster, namely ($x \in \{1 + n_1 + m_1, \ldots, n_1 + m_1 + n_2\}$). A rough estimation of the P_I can be thought of as the product of the probability of finding a *RLA* in the first *a-type* cluster by the probability of realizing $f_a > f_b$ condition when the *RLA* is indeed in the first cluster

$$P_I = \frac{n_1}{N} P(x_0, f_a > f_b) \qquad x_0 \in \{1, 2, \ldots, n_1\}. \tag{29}$$

In a similar manner we may write

$$P_{II} = \frac{n_2}{N} P(x_0^{'}, f_a > f_b) \qquad x_0^{'} \in \{1 + n_1 + m_1, \ldots, n_1 + m_1 + n_2\}. \tag{30}$$

It is necessary to use the symmetry of the configuration to bypass some tremendous calculus by putting $n_1 = m_1 = m_2, n_2 = 2n_1, x_0 = 1, x_0^{'} = 1 + n_1 + m_1$. Let us define the ratio of the previousely defined probabilities

$$R = \frac{P_I}{P_{II}} = \frac{n_1}{n_2} \frac{P(x_0, f_a > f_b)}{P(x_0^{'}, f_a > f_b)}. \tag{31}$$

For $n_1 = n_2$, evidently, $R = 1$, in other words it is equally probable for the first *a-type* cluster to grow as the second one. For $n_1 \neq n_2$ we expect this number to indicate which cluster is more probable to grow. A rough estimation of this ratio can be done using the first approximation of the term $P(x_0, f_a > f_b)$ from (26)

$$R = \frac{P_I}{P_{II}} \cong \frac{n_1}{n_2} \prod_{i=1}^{\tau} \frac{p_i(x_0)}{p_i(x_0^{'})}. \tag{32}$$

With the help of (27) it is now possible to write

$$p_i(x) = \sum_{j=1}^{N} a_{ij}(x) g(j) \tag{33}$$

where $a_{ij}(x = 1)$ are the elements of the first row of the matrix A^i. It is also straightforward that $a_{ij-x}(1) = a_{ij}(x)$, which means that on changing the starting point of the *RLA* from $x_0 = 1$ to $x_0 = x$ a cyclic permutation of the columns of matrix A^i is realized. In view of the preceding statements it follows that

$$\frac{p_i(1)}{p_i(1 + n_1 + m_1)} = \frac{\sum_{j=1}^{N} a_{ij}(1) g(j)}{\sum_{j=1}^{N} a_{ij}(1 + n_1 + m_1) g(j)} = \frac{\sum_{j=1}^{m} a_{ij}(1) + \sum_{j=1+2m}^{4m} a_{ij}(1)}{\sum_{j=1+4m}^{5m} a_{ij}(1) + \sum_{j=1+m}^{3m} a_{ij}(1)}. \tag{34}$$

Taking $p^+ = p^- = 0.5$ then the coefficients a_{ij} are binomial coefficients. We can observe, by direct evaluation from (21), that for big enough values of i the first line of the A^i matrix has the form

$$\binom{k}{2k}\,0\,\binom{k-1}{2k}\,0\,\binom{k-2}{2k}\,0\ldots\binom{k+2}{2k}\,0\,\binom{k+1}{2k}\,0$$

for even i. Using the symmetry of the binomial coefficients $\left(\binom{i}{2k} = \binom{2k-i}{2k}\right)$ we have $\frac{p_i(1)}{p_i(1+n_1+m_1)} \cong 1$. Therefore, the final estimation of the ratio is $R \cong \frac{n_1}{n_2} = 0.5$. Despite the unrealistic simplifications made up to now it was found that the probability of realizing the condition $f_a > f_b$, and therefore of putting down an *a-type object*, is as great as the cluster dimension. In other words, the probability of growing a cluster (pile) is directly proportional with its dimension (i.e. the number of *objects* in it). Therefore the oscillatory phenomena, if they exist, are only transitory. That completes the proof of the convergence. The basic tool of more refined analysis concerning the dependence of R on other parameters $(p^+, p^-, x_0, m_1, m_2)$ remain the relations (26) and (31) and work is in progress.

Finally, we should comment the influence of the initial approximations made to obtain the analytical result (31). First of all we considered in (18) only one *RLA* $F = 1$ case. The assumption is valid if the coupling between the *entities* is very weak. If the concentration of the *objects* in the *environment* is very low the condition is fulfilled. Our numerical simulation, with concentrations of up to 10% (see figure 2), indicates a good agreement with theoretical established Monte Carlo timesteps. Therefore, we may conclude that nonlinear coupling of the *entities'* local dynamics does not play a significant role in the low-concentration limit and our results are valid. We also considered here a near-equilibrium particular configuration (see figure 3) to demonstrate the sufficient convergence condition. This particular choice is based on a numerical observed oscillatory phenomenon which manifests itself near the final aggregation stage. We demonstrate that the probability of growing a cluster with one unit is proportional to its dimension and, therefore, the oscillations are only transient phenomena.

4.1. Numerical simulations

Let us briefly discuss the numerical simulations results that support our conclusion.

One set of simulations was done with $p^+ = p^- = 0.5$ and $x_0 = 1$. We let *RLA* to walk through the lattice ($\tau = 10^6$ steps). Every time when $f_a > f_b$ and the *RLA* is in the first *a-type* cluster the variable N_1 increases by one unit. Otherwise when $f_a > f_b$ and the *RLA* is in the second *a-type* cluster N_2 increases by one unit. It is assumed that the ratio R (see (31)) can be estimated by the ratio $\frac{N_1}{N_2}$. The numerical simulations were performed for constant dimension of the first *a-type* cluster ($n_1 = 10$) and increasing the dimension of the second one ($n_2 = 20$, 40, 60, 80, 100, 120, 140, 160, 180, 200). Figure 4 shows the dependence of the probability ratio $\frac{N_1}{N_2}$ on the two-cluster dimension ratio $\frac{n_1}{n_2}$. As we theoretically predicted, this number must decrease when $\frac{n_1}{n_2}$ decreases. In figure 4 there are three different plots of the $\frac{N_1}{N_2}$ dependence on $\frac{n_1}{n_2}$ for different dimensions of the *b-type* clusters ($m_1 = m_2 = 50$, 100, 500). It can be seen that the influence of the of the *b-type* clusters on the aggregation of the *a-type* clusters is not significant.

Our sufficient convergence demonstration is based on the evaluation of the ratio R (see (31)) which (according to figure 4) confirms the theoretical predicted dependence. But, our previous results were done in a particular case, namely, only one *RLA* in the lattice ($F = 1$). We performed numerical simulations with $F = 30$ *RLA* in a

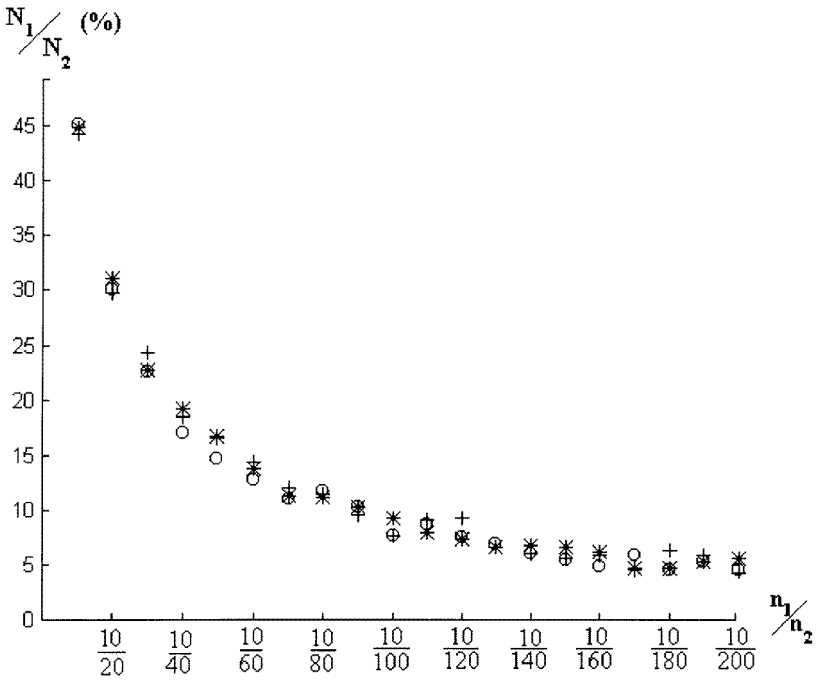

Figure 4. The probability ratio $\frac{N_1}{N_2}$ dependence on the ratio $\frac{n_1}{n_2}$ of the two *a-type* cluster for different *b-type* clusters dimensions ($m_1 = m_2 = m$). '$*$' corresponds to $m = 50$, '$+$' to $m = 100$ and 'o' to $m = 500$. In all these simulations $r = 1.5$ and $x_0 = 1$.

periodic one-dimensional lattice *environment* (see figure 5). Different *a-type* cluster dimensions were used $n_1 = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$ and, respectively, $n_2 = 20, 40, 60, 80, 100, 120, 140, 160, 180, 200$, with a constant ratio of the two *a-type* clusters ($\frac{n_1}{n_2} = \frac{1}{2}$). Here, as we pointed out in the previous case, three different plots of the $\frac{N_1}{N_2}$dependence on $\frac{n_1}{n_2}$ were considered. Comparing the $\frac{N_1}{N_2}$ values from figures 4 and 5 we find that an increase of the number of *entities* does not change the established sufficient convergence condition.

Finally, we performed numerical simulations starting from different initial positions. It can be seen from figure 6 that the mean value of the ratio $\frac{N_1}{N_2}$, with a constant $\frac{n_1}{n_2} = 0.5$ value, remains in the expected range for every initial position x_0.

5. Conclusions

In this paper we analysed the convergence properties of the FSO algorithm.

We found necessary analytical conditions to start *two-object* cluster aggregation. We also found a good agreement between the theoretical and numerical limits of the Monte Carlo timesteps that ensure the convergence of the algorithm.

We also proved that, near the final aggregation stage, the probability of increasing a cluster (pile) is proportional to its dimension (the number of included *objects*). Therefore, we derived a sufficient convergence condition and demonstrate that the algorithm cannot be trapped in an oscillatory regime.

We considered a particular, monotonically decreasing, *weight function*. To include some

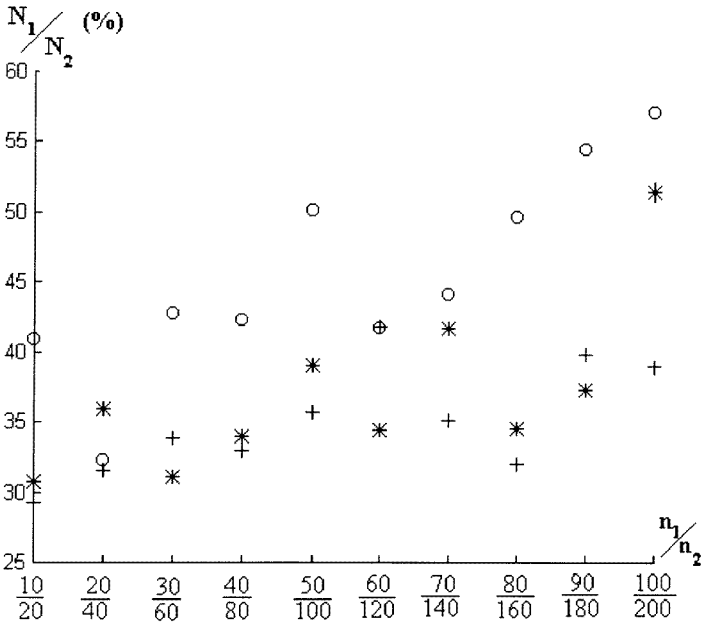

Figure 5. The ratio $\frac{N_1}{N_2}$ dependence on the *b-type* cluster dimension for the fixed $\frac{n_1}{n_2} = 0.5$ value of the two *a-type* clusters. '$*$' corresponds to $m_1 = m_2 = 50$, '$+$' to $m_1 = m_2 = 100$ and 'o' to $m_1 = m_2 = 500$. In all these simulations $r = 1.5$ and $x_0 = 1$.

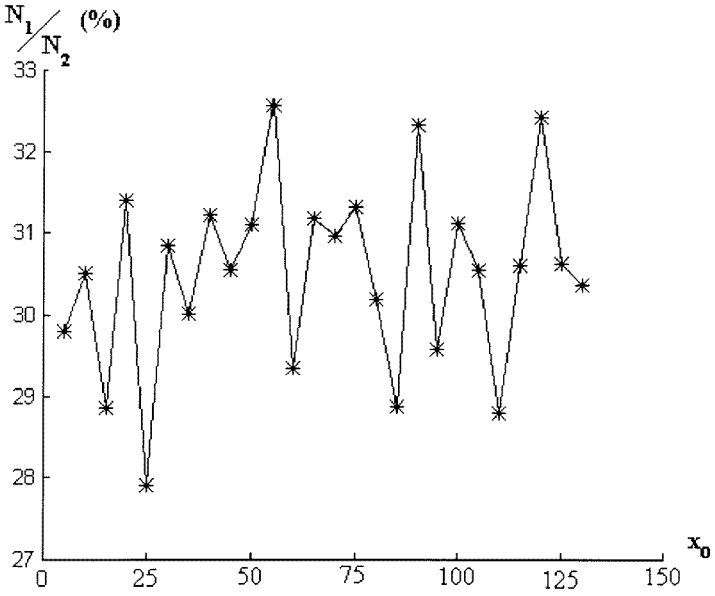

Figure 6. The ratio $\frac{N_1}{N_2}$ dependence on the starting position x_0 in the lattice for the fixed values $\frac{n_1}{n_2} = 0.5$ and $m_1 = m_2 = 100$.

experimental facts related to the self-organized response of the immune system to tumour attack we considered a different-monotonically increasing—*weight function*. The work is in progress and some preliminary results were published (see [11]).

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